

Generalized Fibonacci and Lucas Numbers of the form $5x^2$

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Abstract

Let $(U_n(P, Q))$ and $(V_n(P, Q))$ denote the generalized Fibonacci and Lucas sequence, respectively. In this study, we assume that $Q = 1$. We determine all indices n such that $U_n = 5x^2$ and $U_n = 5U_m x^2$ under some assumptions on P . We show that the equation $V_n = 5x^2$ has the solution only if $n = 1$ for the case when P is odd. Moreover, we show that the equation $V_n = 5V_m x^2$ has no solutions.

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1. Introduction

Let P and Q be non-zero integers with $P^2 + 4Q \neq 0$. Generalized Fibonacci sequence $(U_n(P, Q))$ and Lucas sequence $(V_n(P, Q))$ are defined by the following recurrence relations:

$$U_0(P, Q) = 0, U_1(P, Q) = 1, U_{n+2}(P, Q) = PU_{n+1}(P, Q) + QU_n(P, Q) \text{ for } n \geq 0$$

and

$$V_0(P, Q) = 2, V_1(P, Q) = P, V_{n+2}(P, Q) = PV_{n+1}(P, Q) + QV_n(P, Q) \text{ for } n \geq 0.$$

$U_n(P, Q)$ is called the n -th generalized Fibonacci number and $V_n(P, Q)$ is called the n -th generalized Lucas number. Also generalized Fibonacci and Lucas numbers for negative subscripts are defined as

$$U_{-n}(P, Q) = \frac{-U_n(P, Q)}{(-Q)^n} \text{ and } V_{-n} = \frac{V_n(P, Q)}{(-Q)^n} \text{ for } n \geq 1,$$

respectively. Taking $\alpha = (P + \sqrt{P^2 + 4Q})/2$ and $\beta = (P - \sqrt{P^2 + 4Q})/2$ to be the roots of the characteristic equation $x^2 - Px - Q = 0$, we have the well known expressions named Binet forms

$$U_n(P, Q) = (\alpha^n - \beta^n)/(\alpha - \beta) \text{ and } V(P, Q) = \alpha^n + \beta^n$$

for all $n \in \mathbb{Z}$. From now on, we assume that $P > 0$ and $P^2 + 4Q > 0$. Instead of $U_n(P, Q)$ and $V_n(P, Q)$, we will use U_n and V_n , respectively. For $P = Q = 1$, the sequence (U_n) is the familiar Fibonacci sequence (F_n) and the sequence (V_n) the familiar Lucas sequence (L_n) . If $P = 2$ and $Q = 1$, then we have the well known Pell sequence (P_n) and Pell-Lucas sequence (Q_n) . For $Q = -1$, we represent (U_n) and (V_n) by (u_n) and (v_n) , respectively. For more information about generalized Fibonacci and Lucas sequences, one can consult [1, 2, 3, 4].

Investigations of the properties of second order linear recurring sequences, have given rise to questions concerning whether, for certain pairs (P, Q) , U_n or V_n is square(= \square). In particular, the squares in sequences (U_n) and (V_n) were investigated by many authors. Ljunggren [5] showed in 1942 that if $P = 2$, $Q = 1$, and $n \geq 2$, then $U_n = \square$ precisely for $n = 7$ and $U_n = 2\square$ precisely for $n = 2$. In 1964, Cohn [6] proved that if $P = Q = 1$, then the only perfect square greater than 1 in the sequence (U_n) is $U_{12} = 12^2$ (see also Alfred [7], Burr [8], and Wyler [9]), and he [10, 11] solved the equations $U_n = 2\square$ and $V_n = \square, 2\square$. Furthermore, in other papers, Cohn [12, 13] determined the squares and twice the squares in (U_n) and (V_n) when P is odd and $Q = \pm 1$. Ribenboim and McDaniel [14] determined all indices n such that $U_n = \square$, $2U_n = \square$, $V_n = \square$ or $2V_n = \square$ for all odd relatively prime integers P and Q . In 1998, Kagawa and Terai [15] considered a similar problem for the case when P is even and $Q = 1$. Using the elementary properties of elliptic curves, they showed that if $P = 2t$ with t even and $Q = 1$, then $U_n = \square$, $2U_n = \square$, $V_n = \square$ or $2V_n = \square$ implies $n \leq 3$ under some assumptions. Besides, for $Q = 1$, Nakamura and Petho [16] gave the solutions of the equations $U_n = w\square$ where $w \in \{1, 2, 3, 6\}$. In 1998, Ribenboim and McDaniel [17] showed that if P is even, $Q \equiv 3(\text{mod } 4)$ and $U_n = \square$, then n is a square

or twice an odd square and all prime factors of n divides $P^2 + 4Q$. In a latter paper, the same authors [18] solved the equation $V_n = 3\Box$ for $P \equiv 1, 3 \pmod{8}$, $Q \equiv 3 \pmod{4}$, $(P, Q) = 1$ and solved the equation $U_n = 3\Box$ for all odd relatively prime integers P and Q . Moreover, in [19], Cohn solved the equations $V_n = V_m x^2$ and $V_n = 2V_m x^2$ when P is odd. Keskin and Yosma [20] gave the solutions of the equations $F_n = 2F_m\Box$, $L_n = 2L_m\Box$, $F_n = 3F_m\Box$, $F_n = 6F_m\Box$, and $L_n = 6L_m\Box$. In [27], Şiar and Keskin, assuming $Q = 1$, solve the equation $V_n = 2V_m\Box$ when P is even. They determine all indices n such that $V_n = kx^2$ when $k|P$ and P is odd, where k is a square-free positive divisor of P . They show that there is no integer solution of the equations $V_n = 3\Box$ and $V_n = 6\Box$ for the case when P is odd and also they give the solution of the equations $V_n = 3V_m\Box$ and $V_n = 6V_m\Box$. More generally, we can give the following theorem proved by Shorey and Stewart in [28]:

Let $A > 0$ be an integer. Then there exists an effectively computable number $C > 0$, which depends on A , such that if $n > 0$ and $U_n = A\Box$ or $V_n = A\Box$, then $n < C$.

In this study, we assume, from this point on, that $Q = 1$. We determine all indices n such that $U_n = 5\Box$ and $U_n = 5U_m\Box$ under some assumptions on P . We show that if P is odd, then the equation $V_n = 5\Box$ has the solution only if $n = 1$. Moreover, we prove that the equation $V_n = 5V_m\Box$ has no solutions.

2. Preliminaries

In this section, we give some theorems, lemmas and well known identities about generalized Fibonacci and Lucas numbers, which will be needed in the proofs of the main theorems. Through the paper $\left(\frac{*}{*}\right)$ denotes the Jacobi symbol. The proofs of the following two theorems can be found in [21].

Theorem 2.1. *Let $m, r \in \mathbb{Z}$ and n be non-zero integer. Then*

$$U_{2mn+r} \equiv (-1)^{mn} U_r \pmod{U_m} \quad (2.1)$$

and

$$V_{2mn+r} \equiv (-1)^{mn} V_r \pmod{U_m} \quad (2.2)$$

Theorem 2.2. *Let $m, r \in \mathbb{Z}$ and n be non-zero integer. Then*

$$U_{2mn+r} \equiv (-1)^{(m+1)n} U_r \pmod{V_m} \quad (2.3)$$

and

$$V_{2mn+r} \equiv (-1)^{(m+1)n} V_r \pmod{V_m} \quad (2.4)$$

We state the following theorem from [16].

Theorem 2.3. *Let $P > 0$ and $Q = 1$. If $U_n = wx^2$ with $w \in \{1, 2, 3, 6\}$, then $n \leq 2$ except when $(P, n, w) = (2, 4, 3), (2, 7, 1), (4, 4, 2), (1, 12, 1), (1, 3, 2), (1, 4, 3), (1, 6, 2)$, and $(24, 4, 3)$.*

We give the following two theorems from [12] and [13].

Theorem 2.4. *If P is odd, then the equation $V_n = x^2$ has the solutions $n = 1$, $P = \square$, and $P \neq 1$ or $n = 1, 3$ and $P = 1$ or $n = 3$ and $P = 3$.*

Theorem 2.5. *If P is odd, then the equation $V_n = 2x^2$ has the solutions $n = 0$ or $n = \pm 6$ and $P = 1, 5$.*

The following two theorems can be obtained from Theorem 11 and Theorem 12 given in [19].

Theorem 2.6. *Let P be an odd integer, $m \geq 1$ be an integer and $V_n = V_m x^2$ for some integer x . Then $n = m$.*

Theorem 2.7. *If P is an odd integer and $m \geq 1$, then there is no integer x such that $V_n = 2V_m x^2$.*

The following theorem can be obtained from Theorem 6 given in [19].

Theorem 2.8. *Let P be an odd integer, $m \geq 2$ be an integer and $U_n = 2U_m x^2$ for some integer x . Then $n = 12, m = 6, P = 5$.*

Now we give some well known theorems in number theory. For more detailed information, see [22] or [23].

Theorem 2.9. *Let m be an odd integer. Suppose that $x^2 \equiv -a^2 \pmod{m}$ for some nonzero integers x and a . Then $m \equiv 1 \pmod{4}$.*

We omit the proof of the following theorem since it can be seen easily by induction method.

Theorem 2.10. *Let k be an integer with $k \geq 1$. Then $L_{2^k} \equiv 3 \pmod{4}$.*

Corollary 1. *Let a be any nonzero integer. If $k \geq 1$, then there is no integer x such that $x^2 \equiv -a^2 \pmod{L_{2^k}}$.*

We omit the proof of the following theorem due to Keskin and Demirtürk [24].

Theorem 2.11. *All nonnegative integer solutions of the equation $u^2 - 5v^2 = 1$ are given by $(u, v) = (L_{3z}/2, F_{3z}/2)$ with nonnegative even integer z and all nonnegative integer solutions of the equation $u^2 - 5v^2 = -1$ are given by $(u, v) = (L_{3z}/2, F_{3z}/2)$ with positive odd integer z .*

By using the above theorem, we can give the following theorem without proof.

Theorem 2.12. *All nonnegative integer solutions of the equation $x^2 - 4xy - y^2 = -5$ are given by $(x, y) = (L_{3z+3}/2, L_{3z}/2)$ with nonnegative even integer z and all nonnegative integer solutions of the equation $x^2 - 4xy - y^2 = -1$ are given by $(x, y) = (F_{3z+3}/2, F_{3z}/2)$ with positive odd integer z .*

The following lemma can be found in [18].

Lemma 1. *Let P be odd, m be an odd positive integer, and $r \geq 1$. Then*

$$V_{2^r m} \equiv \begin{cases} 2 \pmod{8} & \text{if } 3 \mid m, \\ 3 \pmod{8} & \text{if } 3 \nmid m \text{ and } r = 1, \\ 7 \pmod{8} & \text{if } 3 \nmid m \text{ and } r > 1. \end{cases}$$

Now we give the following results involving Fibonacci and Lucas numbers with nonnegative integers a and m .

$$F_m = a^2 \text{ iff } m = 0, 1, 2, 12, \tag{2.5}$$

$$F_m = 2a^2 \text{ iff } m = 0, 3, 6, \tag{2.6}$$

$$F_m = 5a^2 \text{ iff } m = 0, 5, \tag{2.7}$$

$$F_m = 10a^2 \text{ iff } m = 0, \tag{2.8}$$

$$L_m = a^2 \text{ iff } m = 1, 3, \quad (2.9)$$

$$L_m = 2a^2 \text{ iff } m = 0, 6. \quad (2.10)$$

The equations (2.5) and (2.6) are Theorems 3 and 4 in [11]; (2.7) follows from Theorem 3 in [25]; (2.8) follows from Theorem 3 in [26]; (2.9) and (2.10) are Theorems 1 and 2 in [11].

We will need the following identities concerning generalized Fibonacci and Lucas numbers:

$$U_{2n} = U_n V_n, \quad (2.11)$$

$$V_{2n} = V_n^2 - 2(-1)^n, \quad (2.12)$$

$$V_n^2 - (P^2 + 4)U_n^2 = 4(-1)^n, \quad (2.13)$$

$$U_{3n} = U_n ((P^2 + 4)U_n^2 + 3(-1)^n), \quad (2.14)$$

$$u_{3n} = u_n ((P^2 - 4)u_n^2 + 3) \quad (2.15)$$

$$U_{5n} = \begin{cases} U_n ((P^2 + 4)^2 U_n^4 + 5(P^2 + 4)U_n^2 + 5) & \text{if } n \text{ is even} \\ U_n ((P^2 + 4)^2 U_n^4 - 5(P^2 + 4)U_n^2 + 5) & \text{if } n \text{ is odd,} \end{cases} \quad (2.16)$$

$$V_{5n} = \begin{cases} V_n (V_n^4 - 5V_n^2 + 5) & \text{if } n \text{ is even} \\ V_n (V_n^4 + 5V_n^2 + 5) & \text{if } n \text{ is odd,} \end{cases} \quad (2.17)$$

$$\text{If } m \geq 1, \text{ then } V_m | V_n \text{ iff } m | n \text{ and } n/m \text{ is odd integer,} \quad (2.18)$$

$$\text{If } U_m \neq 1, \text{ then } U_m | U_n \text{ iff } m | n. \quad (2.19)$$

$$\text{If } P \text{ is odd, then } (U_n, V_n) = \begin{cases} 1 & \text{if } 3 \nmid n \\ 2 & \text{if } 3 \mid n, \end{cases} \quad (2.20)$$

$$\text{If } r \geq 3, \text{ then } V_{2r} \equiv 2 \pmod{V_2}. \quad (2.21)$$

If $5|P$ and n is odd, then $5|V_n$ and therefore from (2.17), it follows that

$$V_{5n} = 5V_n(5a + 1) \quad (2.22)$$

for some positive integer a .

3. Main Theorems

From this point on, we assume that $m, n \geq 1$. Now we prove two theorems which help us to determine for what values of n , the equation $U_n = 5x^2$ has solutions and for what values of m, n , the equations $V_n = 5V_mx^2$ and $U_n = 5U_mx^2$ have solutions.

Theorem 3.1. *The only positive integer solution of the equation $x^4 + 3x^2 + 1 = 5y^2$ is given by $(x, y) = (1, 1)$ and the only positive integer solution of the equation $x^4 - 3x^2 + 1 = 5y^2$ is given by $(x, y) = (2, 1)$.*

Proof. Assume that $x^4 \pm 3x^2 + 1 = 5y^2$ for some positive integers x and y . Multiplying both sides of the equations by 4 and completing the square gives

$$(2x \pm 3)^2 - 5 = 5(2y)^2.$$

Then it follows that

$$(2y)^2 - 5((2x \pm 3)/5)^2 = -1.$$

By Theorem 3.1, we get $2y = L_{3z}/2$ and $(2x^2 \pm 3)/5 = F_{3z}/2$ with positive odd integer z . Assume that $z > 1$. Then we can write $z = 4q \pm 1$ for some $q > 0$ and therefore $z = 2 \cdot 2^k a \pm 1$ with $2 \nmid a$ and $k \geq 1$. Thus by (2.3), we get

$$F_{3z} = F_{3(4q \pm 1)} = F_{12q \pm 3} = F_{2 \cdot 2^k 3a \pm 3} \equiv -F_{\pm 3} \equiv -F_3 \pmod{L_{2^k}},$$

i.e.,

$$F_{3z} \equiv -2 \pmod{L_{2^k}}.$$

Substituting the value of F_{3z} and rewriting the above congruence gives

$$4x^2 \pm 6 \equiv -10 \pmod{L_{2^k}}.$$

This shows that

$$4x^2 + 6 \equiv -10 \pmod{L_{2^k}} \text{ or } 4x^2 - 6 \equiv -10 \pmod{L_{2^k}}.$$

Then it follows that

$$x^2 \equiv -4 \pmod{L_{2^k}}$$

or

$$x^2 \equiv -1 \pmod{L_{2^k}},$$

which is a contradiction by Corollary 1. Thus $z = 1$ and therefore $2x^2 \pm 3 = 5F_3/2$ and $2y = L_3/2$. A simple computation shows that $y = 1$ and $x = 1$ or $x = 2$. This means that the equation $x^4 + 3x^2 + 1 = 5y^2$ has only the positive integer solution $(x, y) = (1, 1)$ and the equation $x^4 - 3x^2 + 1 = 5y^2$ has only the positive integer solution $(x, y) = (2, 1)$. This completes the proof of Theorem 3.1. ■

Theorem 3.2. *The equation $x^4 + 5x^2 + 5 = 5y^2$ has no solution in positive integers x and y .*

Proof. Assume that $x^4 + 5x^2 + 5 = 5y^2$ for some positive integers x and y . Then, since $(2y + 2)^2 + (4y - 1)^2 = 20y^2 + 5$, it follows that

$$(2y + 2)^2 + (4y - 1)^2 = (2x^2 + 5)^2.$$

Clearly, $d = (2y + 2, 4y - 1) = 1$ or 5 . Assume that $d = 1$. By the Pythagorean theorem, there exist positive integers a and b with $(a, b) = 1$, a and b are opposite parity, such that

$$2x^2 + 5 = a^2 + b^2, \quad 2y + 2 = 2ab, \quad 4y - 1 = a^2 - b^2.$$

The latter two equations imply that

$$-5 = a^2 - 4ab - b^2. \tag{3.1}$$

Thus by Theorem 2.12, we get $a = L_{3z+3}/2$, $b = L_{3z}/2$ with nonnegative even integer z . On the other hand, from the equations $-5 = a^2 - 4ab - b^2$ and $2x^2 + 5 = a^2 + b^2$, we readily obtain $x^2 = a(a - 2b)$. Since $(a, b) = 1$, it follows that, $r = (a, a - 2b) = 1$ or 2 . If $r = 1$, then there exist coprime positive integers u and v such that $a = u^2$, $a - 2b = v^2$. Thus $L_{3z+3} = 2a = 2u^2$ and therefore $3z + 3 = 6$ by (2.10), which is impossible since z is even. If $r = 2$, then $a = 2u^2$, $a - 2b = 2v^2$. Thus $L_{3z+3} = 4u^2 = (2u)^2$ and therefore $3z + 3 = 1$ or 3 by (2.9). The first of these is impossible. And the second implies that $z = 0$. Thus $a = 2$, $b = 1$. Since $2x^2 + 5 = a^2 + b^2$, it follows that $x = 0$, which is impossible since x is positive. Assume that $d = 5$. Then there exist positive integers a and b with $(a, b) = 1$, a and b are opposite parity, such that

$$2x^2 + 5 = 5a^2 + 5b^2, \quad 2y + 2 = 10ab, \quad 4y - 1 = 5a^2 - 5b^2.$$

The above first equation implies that $5|x$ and therefore $x = 5t$ for some positive integer t . And the latter two equations imply that $-5 = 5a^2 - 20ab - 5b^2$, i.e.,

$-1 = a^2 - 4ab - b^2$. and completing the square gives $(a - 2b)^2 - 5b^2 = -1$. Thus by Theorem 2.12, we get $a = F_{3z+3}/2$, $b = F_{3z}/2$ with positive odd integer z . On the other hand, by using $x = 5t$, from the equations $-5 = 5a^2 - 20ab - 5b^2$ and $2x^2 + 5 = 5a^2 + 5b^2$, we obtain $5t^2 = a(a - 2b)$. Since $(a, b) = 1$, clearly, $(a, a - 2b) = 1$ or 2 . Assume that $(a, a - 2b) = 1$. This implies that either $a = 5u^2, a - 2b = v^2$ or $a = u^2, a - 2b = 5v^2$. If the first of these is satisfied, then it is seen that $F_{3z+3} = 10u^2$ and therefore $3z + 3 = 0$ by (2.8), which is impossible in positive integers. If the second is satisfied, then it is seen that $F_{3z+3} = 2u^2$ and therefore $3z + 3 = 0, 3$ or 6 by (2.6). But it is obvious that the cases $3z + 3 = 0$ and $3z + 3 = 3$ are impossible in positive integers. If $3z + 3 = 6$, then $z = 1$ and therefore $a = 2, b = 1$. Since $2x^2 + 5 = 5a^2 + 5b^2$, it follows that $x^2 = 10$, which is impossible. Assume that $(a, a - 2b) = 2$. Then either $a = 10u^2, a - 2b = 2v^2$ or $a = 2u^2, a - 2b = 10v^2$. If the first of these is satisfied, then $F_{3z+3} = 20u^2 = 5(2u)^2$ and therefore $3z + 3 = 0$ or 5 by (2.7), which are impossible in positive integers. If the second is satisfied, then $F_{3z+3} = 4u^2 = (2u)^2$ and therefore $3z + 3 = 0, 1, 2$ or 12 by (2.5). But there does not any positive integer z such that $3z + 3 = 0, 1$ or 2 . If $3z + 3 = 12$, then we get $z = 3$ and therefore $a = 72, b = 17$. Since $2x^2 + 5 = 5a^2 + 5b^2$, it follows that $x^2 = 13680$, which is impossible. This completes the proof of Theorem 3.2. ■

We now state the following lemma without proof since its proof can be given by induction method.

Lemma 2. *If n is even, then $V_n \equiv 2 \pmod{P^2}$ and if n is odd, then $V_n \equiv nP \pmod{P^2}$.*

From Lemma 2 and identity (2.13), we can give the following corollary.

Corollary 2. *$5|V_n$ if and only if $5|P$ and n is odd.*

The proof of the following lemma can be seen from identity (2.21).

Lemma 3. *If P is odd and $r \geq 1$, then $\left(\frac{P^2 + 3}{V_{2^r}}\right) = 1$.*

Theorem 3.3. *If P is odd, then the equation $V_n = 5x^2$ has solutions only if $n = 1$.*

Proof. Assume that $V_n = 5x^2$. Then by Corollary 2, it follows that $5|P$ and n is odd. Assume that $n > 3$. Then we can write $n = 4q + 1$ or $n = 4q + 3$ for some $q \geq 1$. From this point on, we divide the proof into two cases.

Case 1 : Assume that $n = 4q + 1$. Then we can write $n = 4q + 1 = 2(2^k a) + 1$ for some odd integer a with $k \geq 1$. And so by (2.4), we get

$$V_n = V_{2 \cdot 2^k a + 1} \equiv -V_1 \pmod{V_{2^k}},$$

which implies that

$$5x^2 \equiv -P \pmod{V_{2^k}}.$$

Therefore the Jacobi symbol $J = \left(\frac{-5P}{V_{2^k}} \right) = 1$. Assume that $P \equiv 5, 7 \pmod{8}$. Since $V_{2^k} \equiv 2 \pmod{P}$ by Lemma 2, it is seen that $V_{2^k} \equiv 2 \pmod{5}$. This shows that

$$\left(\frac{5}{V_{2^k}} \right) = \left(\frac{V_{2^k}}{5} \right) = \left(\frac{2}{5} \right) = (-1)^{\frac{5^2-1}{8}} = -1$$

and

$$\left(\frac{P}{V_{2^k}} \right) = (-1)^{\left(\frac{P-1}{2}\right)\left(\frac{V_{2^k}-1}{2}\right)} \left(\frac{V_{2^k}}{P} \right) = (-1)^{\left(\frac{P-1}{2}\right)} \left(\frac{2}{P} \right) = (-1)^{\left(\frac{P-1}{2}\right)} (-1)^{\left(\frac{P^2-1}{8}\right)} = -1$$

since $P \equiv 5, 7 \pmod{8}$. Also we have $\left(\frac{-1}{V_{2^k}} \right) = -1$ by Lemma 1. Hence we get

$J = \left(\frac{-5P}{V_{2^k}} \right) = -1$, which contradicts with the fact that $J = 1$. Assume that $P \equiv 1, 3 \pmod{8}$. If we write $n = 4q + 1 = 4(q + 1) - 3 = 2(2^k a) - 3$ for some odd integer a with $k \geq 1$, then we get

$$V_n = V_{2 \cdot 2^k a - 3} \equiv -V_{-3} \equiv V_3 \pmod{V_{2^k}},$$

which implies that

$$5x^2 \equiv V_3 \pmod{V_{2^k}}$$

by (2.4). This shows that $\left(\frac{5V_3}{V_{2^k}} \right) = 1$. Since $V_{2^k} \equiv 2 \pmod{P}$, we get $V_{2^k} \equiv 2 \pmod{5}$ by Lemma 2. Moreover, $\left(\frac{P^2 + 3}{V_{2^k}} \right) = 1$ by Lemma 3 and $V_{2^k} \equiv 3, 7 \pmod{8}$ by

Lemma 1. Then it follows that

$$\begin{aligned}
1 &= \left(\frac{5V_3}{V_{2^k}} \right) = \left(\frac{5}{V_{2^k}} \right) \left(\frac{P}{V_{2^k}} \right) \left(\frac{P^2+3}{V_{2^k}} \right) = \left(\frac{V_{2^k}}{5} \right) (-1)^{\left(\frac{P-1}{2}\right)\left(\frac{V_{2^k}-1}{2}\right)} \left(\frac{V_{2^k}}{P} \right) \\
&= \left(\frac{2}{5} \right) (-1)^{\left(\frac{P-1}{2}\right)} \left(\frac{2}{P} \right) = (-1)(-1)^{\left(\frac{P-1}{2}\right)}(-1)^{\left(\frac{P^2-1}{8}\right)} = -1,
\end{aligned}$$

a contradiction.

Case 2 : Assume that $n = 4q + 3$. We can write $n = 4q + 3 = 2(2^k a) + 3$ for some odd integer a with $k \geq 1$. And so by (2.4), we get

$$V_n = V_{2 \cdot 2^k a + 3} \equiv -V_3 \pmod{V_{2^k}}$$

i.e.,

$$5x^2 = -V_3 \pmod{V_{2^k}}.$$

This shows that $J = \left(\frac{-5V_3}{V_{2^k}} \right) = 1$. Assume that $P \equiv 5, 7 \pmod{8}$. Since $V_{2^k} \equiv 2 \pmod{P}$ by Lemma 2, it is seen that $V_{2^k} \equiv 2 \pmod{5}$. Also we have $\left(\frac{-1}{V_{2^k}} \right) = -1$ by Lemma 1 and $\left(\frac{P^2+3}{V_{2^k}} \right) = 1$ by Lemma 3. Hence we get

$$\begin{aligned}
\left(\frac{-5V_3}{V_{2^k}} \right) &= \left(\frac{-1}{V_{2^k}} \right) \left(\frac{5}{V_{2^k}} \right) \left(\frac{V_3}{V_{2^k}} \right) = \left(\frac{-1}{V_{2^k}} \right) \left(\frac{5}{V_{2^k}} \right) \left(\frac{P}{V_{2^k}} \right) \left(\frac{P^2+3}{V_{2^k}} \right) \\
&= (-1) \left(\frac{V_{2^k}}{5} \right) (-1)^{\left(\frac{P-1}{2}\right)\left(\frac{V_{2^k}-1}{2}\right)} \left(\frac{V_{2^k}}{P} \right) = (-1) \left(\frac{2}{5} \right) (-1)^{\left(\frac{P-1}{2}\right)} \left(\frac{2}{P} \right) \\
&= (-1)(-1)(-1)^{\left(\frac{P-1}{2}\right)}(-1)^{\left(\frac{P^2-1}{8}\right)} = -1
\end{aligned}$$

since $P \equiv 5, 7 \pmod{8}$. This contradicts with the fact that $J = 1$. Assume that $P \equiv 1, 3 \pmod{8}$. If we write $n = 4q + 3 = 4(q+1) - 1 = 2(2^k a) - 1$ for some odd integer a with $k \geq 1$, then we get

$$V_n = V_{2 \cdot 2^k a - 1} \equiv -V_{-1} \equiv V_1 \pmod{V_{2^k}}$$

i.e.,

$$5x^2 = P \pmod{V_{2^k}}.$$

This shows that $\left(\frac{5P}{V_{2^k}}\right) = 1$. Since $V_{2^k} \equiv 2 \pmod{P}$, we get $V_{2^k} \equiv 2 \pmod{5}$ by Lemma 2. Then it follows that

$$\begin{aligned} 1 &= \left(\frac{5P}{V_{2^k}}\right) = \left(\frac{5}{V_{2^k}}\right) \left(\frac{P}{V_{2^k}}\right) = \left(\frac{V_{2^k}}{5}\right) (-1)^{\left(\frac{P-1}{2}\right)\left(\frac{V_{2^k}-1}{2}\right)} \left(\frac{V_{2^k}}{P}\right) \\ &= \left(\frac{2}{5}\right) (-1)^{\left(\frac{P-1}{2}\right)} \left(\frac{2}{P}\right) = (-1)(-1)^{\left(\frac{P-1}{2}\right)} (-1)^{\left(\frac{P^2-1}{8}\right)} = -1, \end{aligned}$$

a contradiction. We conclude that $n = 1$ or $n = 3$. If $n = 3$, then $V_3 = P(P^2 + 3) = 5x^2$. Since $5|P$, it follows that $(P/5)(P^2 + 3) = x^2$. Clearly, $d = (P/5, P^2 + 3) = 1$ or 3 . Assume that $d = 1$. This implies that $P = 5a^2$ and $P^2 + 3 = b^2$ for some positive integers a and b . Since $5|P$, we get $b^2 \equiv 3 \pmod{5}$, which is impossible. Assume that $d = 3$. Then we get $P = 15a^2$ and $P^2 + 3 = 3b^2$ for some positive integers a and b . It is seen from $P^2 + 3 = 3b^2$ that $3|P$ and therefore $P = 3c$ for some positive integer c . Hence we obtain the Pell equation $b^2 - 3c^2 = 1$. It is well known that all positive integer solutions of this equation are given by $(b, c) = (v_m(4, -1)/2, u_m(4, -1))$ with $m \geq 1$. On the other hand, if we substitute the value $P = 15a^2$ into $P = 3c$, we get $c = 5a^2$. So we are interested in whether the equation $5\Box = u_m(4 - 1)$ has a solution. Assume that the equation $5\Box = u_m(4 - 1)$ has a solution. Since $5|u_3$, it can be seen that if $5|u_m$, then $3|m$ and therefore $m = 3r$ for some positive integer r . Thus from (2.15) we get $u_m = u_{3r} = u_r((P^2 - 4)u_r^2 + 3) = u_r(12u_r^2 + 3)$. Clearly, $(u_r, 12u_r^2 + 3) = 1$ or 3 . Assume that $(u_r, 12u_r^2 + 3) = 1$. This implies that either $u_r = a^2$, $12u_r^2 + 3 = 5b^2$ or $u_r = 5a^2$, $12u_r^2 + 3 = b^2$ for some positive integers a and b . But both of the previous equations are impossible since $b^2 \equiv 3 \pmod{5}$. Assume that $(u_r, 12u_r^2 + 3) = 3$. Then either

$$u_r = 3a^2, \quad 12u_r^2 + 3 = 15b^2 \tag{3.2}$$

or

$$u_r = 15a^2, \quad 12u_r^2 + 3 = 3b^2. \tag{3.3}$$

Assume that (3.2) is satisfied. A simple computation shows that $(2u_r)^2 - 5b^2 = -1$. Thus by Theorem 2.11, we obtain $2u_r = L_{3z}/2$ for some positive odd integer z . Substituting the value $u_r = 3a^2$ into the previous equation gives $3u_r = L_{3z}/4$, i.e., $L_2u_r = L_{3z}/4$. This implies that $L_2|L_{3z}$. Then by (2.18), we get $2|3z$, which is impossible since z is odd. Assume that (3.3) is satisfied. It is easily seen that $(2u_r)^2 + 1 = b^2$, that is, $b^2 - (2u_r)^2 = 1$, implying that $u_r = 0$. This is impossible since r is a positive integer. So $n = 3$ can not be a solution. If $n = 1$, then

$V_1 = P = 5x^2$. It is obvious that this is a solution. This completes the proof of Theorem 3.3. ■

Theorem 3.4. *There is no integer x such that $V_n = 5V_mx^2$.*

Proof. Assume that $V_n = 5V_mx^2$. Then by Corollary 2, it follows that $5|P$ and n is odd. Moreover, since $V_m|V_n$, there exists an odd integer t such that $n = mt$ by (2.18). Thus m is odd. Therefore we have $V_n \equiv nP \pmod{P^2}$ and $V_m \equiv mP \pmod{P^2}$ by Lemma 2. This shows that $nP \equiv 5mPx^2 \pmod{P^2}$, i.e., $n \equiv 5mx^2 \pmod{P}$. Since $5|P$, it follows that $5|n$. Also since $n = mt$, first, assume that $5|t$. Then $t = 5s$ for some positive odd integer s and therefore $n = mt = 5ms$. By (2.17), we readily obtain $V_n = V_{5ms} = V_{ms}(V_{ms}^4 + 5V_{ms}^2 + 5)$. Since ms is odd and $5|P$, it follows that $5|V_{ms}$ by Corollary 2 and therefore $(V_{ms}/V_m)((V_{ms}^4 + 5V_{ms}^2 + 5)/5) = x^2$. Clearly, $(V_{ms}/V_m, (V_{ms}^4 + 5V_{ms}^2 + 5)/5) = 1$. This implies that $V_{ms} = V_ma^2$ and $V_{ms}^4 + 5V_{ms}^2 + 5 = 5b^2$ for some positive integers a and b . Then by Theorem 3.2, we get $V_{ms} = 0$, which is a contradiction. Now assume that $5 \nmid t$. Since $n = mt$ and $5|n$, it is seen that $5|m$. Then we can write $m = 5^r a$ with $5 \nmid a$ and $r \geq 1$. By (2.22), we obtain $V_m = V_{5^r a} = 5V_{5^{r-1}a}(5a_1 + 1)$ for some positive integer a_1 . And thus we conclude that $V_m = V_{5^r a} = 5^r V_a(5a_1 + 1)(5a_2 + 1)\dots(5a_r + 1)$ for some positive integers a_i with $1 \leq i \leq r$. Let $A = (5a_1 + 1)(5a_2 + 1)\dots(5a_r + 1)$. It is obvious that $5 \nmid A$. Thus we have $V_m = 5^r V_a A$. In a similar manner, we see that $V_n = V_{5^r at} = 5^r V_{at}(5b_1 + 1)(5b_2 + 1)\dots(5b_r + 1)$ for some positive integers b_j with $1 \leq j \leq r$. Let $B = (5b_1 + 1)(5b_2 + 1)\dots(5b_r + 1)$. It is obvious that $5 \nmid B$. Thus we have $V_n = 5^r V_{at} B$. This shows that $5^r V_{at} B = 5 \cdot 5^r V_a A x^2$, i.e., $V_{at} B = 5V_a A x^2$. By Lemma 2 and Corollary 2, it is seen that $atPB \equiv 5aPAx^2 \pmod{P^2}$ and therefore we get $atB \equiv 5aAx^2 \pmod{P}$. Since $5|P$, it follows that $5|atB$. But this is impossible since $5 \nmid a, 5 \nmid t$, and $5 \nmid B$. This completes the proof of Theorem 3.4. ■

The following lemma can be proved by using Theorem 2.1.

Lemma 4.

$$5|U_n \Leftrightarrow \begin{cases} 2|n \text{ if } 5|P, \\ 3|n \text{ if } P^2 \equiv -1 \pmod{5}, \\ 5|n \text{ if } P^2 \equiv 1 \pmod{5}, \end{cases}$$

and

$$3|U_n \Leftrightarrow \begin{cases} 2|n \text{ if } 3|P, \\ 4|n \text{ if } 3 \nmid P. \end{cases}$$

Theorem 3.5. *If P is odd and $5|P$, then the equation $U_n = 5x^2$ has the solution $n = 2, P = 5\Box$. If $P^2 \equiv 1 \pmod{5}$, then the equation $U_n = 5x^2$ has the solution $n = 5, P = 1$. If P is odd and $P^2 \equiv -1 \pmod{5}$, then the equation $U_n = 5x^2$ has no solutions.*

Proof. Assume that $5|P$ and P is odd. Since $5|U_n$, it follows that n is even by Lemma 4. Then $n = 2t$ for some positive integer t . By (2.11), we get $U_n = U_{2t} = U_t V_t = 5x^2$. Clearly, $(U_t, V_t) = 1$ or 2 by (2.20). Let $(U_t, V_t) = 1$. This implies that either

$$U_t = a^2, V_t = 5b^2 \quad (3.4)$$

or

$$U_t = 5a^2, V_t = b^2 \quad (3.5)$$

for some positive integers a and b . Assume that (3.4) is satisfied. Since $5|V_t$, it follows that t is an odd integer by Corollary 2. Assume that $t > 1$. Then $t = 4q \pm 1$ for some $q > 1$. We can write $t = 4q \pm 1 = 2 \cdot 2^k u \pm 1$ for some odd integer u with $k \geq 1$. And so by (2.3), we get

$$U_t = U_{2 \cdot 2^k u \pm 1} \equiv -U_{\pm 1} \pmod{V_{2^k}}$$

which implies that

$$a^2 \equiv -1 \pmod{V_{2^k}}.$$

This shows that $1 = \left(\frac{-1}{V_{2^k}}\right)$. But this is impossible since $\left(\frac{-1}{V_{2^k}}\right) = -1$ by Lemma 1. Thus $t = 1$ and therefore $n = 2$. Then $P = 5\Box$ is a solution. Assume that (3.5) is satisfied. Since $5|U_t$, it follows that t is even by Lemma 4. Thus $t = 2r$ for some positive integer r . By using (2.12), we get $V_{2r} = V_r^2 \pm 2 = b^2$, which is impossible. Thus $t = 1$ and therefore $n = 2$. Let $d = 2$. This implies that either

$$U_t = 10a^2, V_t = 2b^2 \quad (3.6)$$

or

$$U_t = 2a^2, V_t = 10b^2 \quad (3.7)$$

for some positive integers a and b . Assume that (3.6) is satisfied. By Theorem 2.5, we have $t = 6$ and $P = 5$. But this is impossible since there does not exist any integer a such that $U_6 = 3640 = 10a^2$. Assume that (3.7) is satisfied. Since $5|V_t$, it follows that t is an odd integer by Corollary 2. If $t = 1$, then $U_1 = 1 = 2a^2$,

which is impossible. Assume that $t > 1$. Then $t = 4q \pm 1$ for some $q > 1$. And so by (2.1), we get

$$U_t = U_{2.2q \pm 1} \equiv U_{\pm 1} \pmod{U_2},$$

implying that

$$2a^2 \equiv 1 \pmod{P}.$$

Since $5|P$, the above congruence becomes

$$2a^2 \equiv 1 \pmod{5},$$

which is impossible since $\left(\frac{2}{5}\right) = -1$. The proof is completed for the case when $5|P$ and P is odd.

Assume that $P^2 \equiv 1 \pmod{5}$. Since $5|U_n$, it follows that $5|n$ by Lemma 4. Thus $n = 5t$ for some positive integer t . Since $P^2 \equiv 1 \pmod{5}$, it is obvious that $5|P^2 + 4$ and therefore there exists a positive integer A such that $P^2 + 4 = 5A$. By (2.16), we get $U_n = U_{5t} = U_t((P^2 + 4)^2 U_t^4 \pm 5(P^2 + 4)U_t^2 + 5)$. Substituting $P^2 + 4 = 5A$ into the previous equation gives $U_n = U_{5t} = 5U_t(5A^2 U_t^4 \pm 5A U_t^2 + 1)$. Let $B = A^2 U_t^4 \pm A U_t^2$. Then we get

$$U_n = U_{5t} = 5U_t(5B + 1) = 5x^2$$

i.e.,

$$U_t(5B + 1) = x^2.$$

It can be seen that $(U_t, 5B + 1) = 1$. This shows that $U_t = a^2$ and $5B + 1 = b^2$ for some positive integers a and b . By Theorem 2.3, we get $t \leq 2$ or $t = 12$ and $P = 1$. If $t = 1$, then $n = 5$ and therefore we get $U_5 = P^4 + 3P^2 + 1 = 5x^2$. By Theorem 3.1, it follows that $P = 1$. So the equation $U_n = 5x^2$ has the solution $n = 5$ and $P = 1$. If $t = 2$, then $n = 10$ and therefore we obtain $U_{10} = 5x^2$, implying that $U_5 V_5 = 5x^2$ by (2.11). Since $5|U_5$, it follows that $(U_5/5)V_5 = x^2$. By (2.20), clearly, $(U_5/5, V_5) = 1$. This implies that $U_5 = 5a^2$, $V_5 = b^2$, which is impossible by Theorem 2.4. If $t = 12$ and $P = 1$, then it follows that $n = 60$. Thus we obtain $U_{60} = 5x^2$, which is impossible by (2.7). The proof is completed for the case when $P^2 \equiv 1 \pmod{5}$.

Assume that $P^2 \equiv -1 \pmod{5}$ and P is odd. Since $5|U_n$, it follows that $3|n$ by Lemma 4 and therefore $n = 3m$ for some positive integer m . Assume that m is even. Then $m = 2s$ for some positive integer s and therefore $n = 6s$. Thus by

(2.11), we get $U_n = U_{6s} = U_{3s}V_{3s} = 5x^2$. By (2.20), clearly, $(U_{3s}, V_{3s}) = 2$. Then either

$$U_{3s} = 10a^2, V_{3s} = 2b^2 \quad (3.8)$$

or

$$U_{3s} = 2a^2, V_{3s} = 10b^2 \quad (3.9)$$

for some positive integer a and b . Assume that (3.8) is satisfied. By Theorem 2.5, it follows that $3s = 6$ and $P = 1, 5$. But this is impossible since $P^2 \equiv -1 \pmod{5}$. Assume that (3.9) is satisfied. Since $5|V_{3s}$, it follows that $5|P$ by Corollary 2. But this contradicts with the fact that $P^2 \equiv -1 \pmod{5}$. Now assume that m is odd. Then by (2.14), we get $U_n = U_{3m} = U_m((P^2 + 4)U_m^2 - 3)$. Clearly, $(U_m, (P^2 + 4)U_m^2 - 3) = 1$ or 3 . Since m is odd, it follows that $3 \nmid U_m$ by Lemma 4 and therefore $(U_m, (P^2 + 4)U_m^2 - 3) = 1$. Then

$$U_m = 5a^2, (P^2 + 4)U_m^2 - 3 = b^2 \quad (3.10)$$

or

$$U_m = a^2, (P^2 + 4)U_m^2 - 3 = 5b^2 \quad (3.11)$$

for some positive integers a and b . Assume that (3.10) is satisfied. Since m is odd, we obtain $V_m^2 + 1 = b^2$ by (2.13). This shows that $V_m = 0$, which is impossible. Assume that (3.11) is satisfied. Since m and P is odd, it follows that $m = 1$ by Theorem 2.3. If $m = 1$, then $n = 3$ and therefore $P^2 + 1 = 5y^2$, which is impossible since we get $y^2 \equiv 2 \pmod{8}$ in this case. This completes the proof of Theorem 3.5. ■

Since the proof of the following lemma can be given by induction method, we omit its proof.

Lemma 5. *If n is even, then $U_n \equiv \frac{n}{2}P \pmod{P^2}$ and if n is odd, then $U_n \equiv 1 \pmod{P^2}$.*

Theorem 3.6. *The equation $U_n = 5U_mx^2$ has no solutions when $P^2 \equiv 1 \pmod{5}$. If P is odd or $4|P$, then the equation $U_n = 5U_mx^2$ has no solutions when $P^2 \equiv -1 \pmod{5}$ and n is odd. If n is even and P is odd, then the equation $U_n = 5U_mx^2$ has no solutions when $P^2 \equiv -1 \pmod{5}$. If P is odd and $5|P$, then the equation $U_n = 5U_mx^2$ has no solutions.*

Proof. Assume that $U_n = 5U_mx^2$ for some positive integer x . If $m = 1$, then $U_n = 5x^2$ which has solutions only if $n = 2$ by Theorem 3.5. So assume that

$m > 1$. Since $U_m | U_n$, it follows that $m | n$ by (2.19). Thus $n = mt$ for some positive integer t . Since $n \neq m$, we have $t > 1$.

Assume that $P^2 \equiv 1 \pmod{5}$. It is obvious that $5 | P^2 + 4$. Since $5 | U_n$, it follows that $5 | n$ by Lemma 4. Now we divide the proof into two cases.

Case 1 : Assume that $5 | t$. Then $t = 5s$ for some positive integer s and therefore $n = mt = 5ms$. By (2.16), we obtain

$$U_n = U_{5ms} = U_{ms} ((P^2 + 4)^2 U_{ms}^4 \pm 5(P^2 + 4)U_{ms}^2 + 5) = 5U_m x^2. \quad (3.12)$$

It is easily seen that $5 | (P^2 + 4)^2 U_{ms}^4 \pm 5(P^2 + 4)U_{ms}^2 + 5$. Also we have $(P^2 + 4)^2 U_{ms}^4 \pm 5(P^2 + 4)U_{ms}^2 + 5 = V_{ms}^4 \pm 3V_{ms}^2 + 1$ by (2.13). So rearranging the equation (3.12) gives

$$x^2 = (U_{ms}/U_m) ((V_{ms}^4 \pm 3V_{ms}^2 + 1)/5).$$

Clearly, $(U_{ms}/U_m, (V_{ms}^4 \pm 3V_{ms}^2 + 1)/5) = 1$. This implies that $U_{ms} = U_m a^2$ and $V_{ms}^4 \pm 3V_{ms}^2 + 1 = 5b^2$ for some positive integers a and b . Thus by Theorem 3.1, we get $V_{ms} = 1$ or $V_{ms} = 2$. The first of these is impossible. If the second is satisfied, then $ms = 0$, which is a contradiction since $m > 1$.

Case 2 : Assume that $5 \nmid t$. Since $5 | n$, it follows that $5 | m$. Then we can write $m = 5^r a$ with $5 \nmid a$ and $r \geq 1$. Since $5 | P^2 + 4$, it can be seen by (2.16) that $U_m = U_{5^r a} = 5U_{5^{r-1}a}(5a_1 + 1)$ for some positive integer a_1 . And thus we conclude that $U_m = U_{5^r a} = 5^r U_a (5a_1 + 1)(5a_2 + 1) \dots (5a_r + 1)$ for some positive integers a_i with $1 \leq i \leq r$. Let $A = (5a_1 + 1)(5a_2 + 1) \dots (5a_r + 1)$. It is obvious that $5 \nmid A$ and we have $U_m = 5^r U_a A$. In a similar manner, we get $U_n = U_{5^r at} = 5^r U_{at} (5b_1 + 1)(5b_2 + 1) \dots (5b_r + 1)$ for some positive integers b_j with $1 \leq j \leq r$. Let $B = (5b_1 + 1)(5b_2 + 1) \dots (5b_r + 1)$. It is obvious that $5 \nmid B$. Thus we have $U_n = 5^r U_{at} B$. Substituting the new values of U_n and U_m into $U_n = 5U_m x^2$ gives

$$5^r U_{at} B = 5 \cdot 5^r U_a A x^2.$$

This shows that

$$U_{at} B = 5U_a A x^2.$$

Since $5 \nmid B$, it follows that $5 | U_{at}$, implying that $5 | at$ by Lemma 4. This contradicts with the fact that $5 \nmid a$ and $5 \nmid t$.

Assume that $P^2 \equiv -1 \pmod{5}$ and n is odd. Then, both m and t are odd. Thus we can write $t = 4q \pm 1$ for some $q \geq 1$. And so by (2.1), we get

$$U_n = U_{(4q \pm 1)m} = U_{2.2mq \pm m} \equiv U_m \pmod{U_{2m}}.$$

This shows that

$$5U_mx^2 \equiv U_m \pmod{U_{2m}}.$$

By using (2.11), we obtain

$$5x^2 \equiv 1 \pmod{V_m}.$$

Since m is odd, it follows that $P|V_m$ by Lemma 2. Then the above congruence becomes

$$5x^2 \equiv 1 \pmod{P}. \quad (3.13)$$

Assume that P is odd. Then (3.13) implies that $J = \left(\frac{5}{P}\right) = 1$. Since $P^2 \equiv -1 \pmod{5}$, it can be seen that $P \equiv \pm 2 \pmod{5}$. Hence we get

$$1 = \left(\frac{5}{P}\right) = \left(\frac{P}{5}\right) = \left(\frac{\pm 2}{5}\right) = -1,$$

a contradiction. Now assume that P is even. If $8|P$, then it follows from (3.13) that $5x^2 \equiv 1 \pmod{8}$, which is impossible since we get $x^2 \equiv 5 \pmod{8}$ in this case. If $4|P$ and $8 \nmid P$, then from (3.13), we get

$$5x^2 \equiv 1 \pmod{P/4}.$$

This shows that $\left(\frac{5}{P/4}\right) = 1$. Since $P^2 \equiv -1 \pmod{5}$, it can be seen that $P/4 \equiv \pm 2 \pmod{5}$. Hence we get

$$1 = \left(\frac{5}{P/4}\right) = \left(\frac{P/4}{5}\right) = \left(\frac{\pm 2}{5}\right) = -1,$$

a contradiction.

Now assume that $P^2 \equiv -1 \pmod{5}$, P is odd, and n is even. Since $n = mt$, we divide the proof into two cases.

Case 1 : Assume that t is even. Then $t = 2s$ for some positive integer s . Thus we get $5x^2 = U_n/U_m = U_{2ms}/U_m = (U_{ms}/U_m)V_{ms}$. Clearly, $d = (U_{ms}/U_m, V_{ms}) = 1$ or 2 by (2.20). Let $d = 1$. Then either

$$U_{ms} = U_ma^2 \text{ and } V_{ms} = 5b^2 \quad (3.14)$$

or

$$U_{ms} = 5U_ma^2 \text{ and } V_{ms} = b^2. \quad (3.15)$$

Assume that (3.14) is satisfied. Since $5|V_{ms}$, it follows that $5|P$ by Corollary 2. This contradicts with the fact that $P^2 \equiv -1 \pmod{5}$. Assume that (3.15) is satisfied. By Theorem 2.4, we get $ms = 3$ and $P = 3$. Since $m > 1$, it follows that $m = 3$. This is impossible since we get $1 = 5a^2$ in this case.

Let $d = 2$. This implies that either

$$U_{ms} = 2U_ma^2 \text{ and } V_{ms} = 10b^2 \quad (3.16)$$

or

$$U_{ms} = 10U_ma^2 \text{ and } V_{ms} = 2b^2. \quad (3.17)$$

Assume that (3.16) is satisfied. Since $5|V_{ms}$, it follows that $5|P$ by Corollary 2. This contradicts with the fact that $P^2 \equiv -1 \pmod{5}$. Assume that (3.17) is satisfied. By Theorem 2.5, we get $ms = 6$ and $P = 1, 5$. But this is impossible since $P^2 \equiv -1 \pmod{5}$.

Case 2 : Assume that t is odd. Since n is even, it follows that m is even. Then there exists a positive integer s such that $m = 2s$. Thus we readily obtain $5x^2 = (U_{st}/U_s)(V_{st}/V_s)$. Clearly, $d = (U_{st}/U_s, V_{st}/V_s) = 1$ or 2 by (2.20). Let $d = 1$. Then either $U_{st} = U_sa^2$ and $V_{st} = 5V_sb^2$ or $U_{st} = 5U_sa^2$ and $V_{st} = V_sb^2$ for some positive integers a and b . The first of these is impossible by Theorem 3.4. If the second is satisfied, then we get $st = s$ by Theorem 2.6. But this is impossible since there does not exist any integer a such that $1 = 5a^2$. Let $d = 2$. This implies that either $U_{st} = 2U_sa^2$ and $V_{st} = 10V_sb^2$ or $U_{st} = 10U_sa^2$ and $V_{st} = 2V_sb^2$ for some positive integers a and b . If the first of these is satisfied, then it follows that $5|V_{st}$. This implies that $5|P$ by Corollary 2, which contradicts with the fact that $P^2 \equiv -1 \pmod{5}$. The second is impossible by Theorem 2.7.

Now assume that $5|P$ and P is odd. Since $5|U_n$, it follows that n is even by Lemma 4. Moreover, since $U_m|U_n$, there exists an integer t such that $n = mt$ by (2.19). Assume that t is even. Then $t = 2s$ for some positive integer s . By (2.11), we get $U_n = U_{2ms} = U_{ms}V_{ms} = 5U_mx^2$, implying that $(U_{ms}/U_m)V_{ms} = 5x^2$. Clearly, $(U_{ms}/U_m, V_{ms}) = 1$ or 2 by (2.20). If $(U_{ms}/U_m, V_{ms}) = 1$, then

$$U_{ms} = U_ma^2, \quad V_{ms} = 5b^2 \quad (3.18)$$

or

$$U_{ms} = 5U_ma^2, \quad V_{ms} = b^2 \quad (3.19)$$

for some positive integers a and b . Assume that (3.18) is satisfied. Then by Theorem 3.3, we get $ms = 1$. This contradicts with the fact that $m > 1$. Assume

that (3.19) is satisfied. Then by Theorem 2.4, we have $ms = 3$ and $P = 1$ or $ms = 3$ and $P = 3$. But both of these are impossible since $5|P$. If $(U_{ms}/U_m, V_{ms}) = 2$, then

$$U_{ms} = 2U_m a^2, \quad V_{ms} = 10b^2 \quad (3.20)$$

or

$$U_{ms} = 10U_m a^2, \quad V_{ms} = 2b^2 \quad (3.21)$$

for some positive integers a and b . Assume that (3.20) is satisfied. Then by Theorem 2.8, we get $ms = 12$, $m = 6$, $P = 5$. On the other hand, since $5|V_{ms}$, it follows by Corollary 2 that $5|P$ and ms is odd. This is a contradiction since $ms = 12$. Assume that (3.21) is satisfied. Then by Theorem 2.5, we have $ms = 6$ and $P = 5$. Since $m > 1$, it is seen that $m = 2, 3$ or 6 . If $m = 2$, then $U_6 = 3640 = 10U_2 x^2 = 50x^2$, i.e., $364 = 5x^2$, which is impossible. If $m = 3$, then $U_6 = 3640 = 10U_3 x^2 = 260x^2$, i.e., $14 = x^2$, which is impossible. If $m = 6$, then there does not exist any integer x such that $1 = 5x^2$.

Now assume that t is odd. Since $n = mt$ and n is even, it follows that m is even. Therefore we have $U_n \equiv (n/2)P \pmod{P^2}$ and $U_m \equiv (m/2)P \pmod{P^2}$ by Lemma 5. This shows that $(n/2)P \equiv 5(m/2)Px^2 \pmod{P^2}$, i.e., $(n/2) \equiv 5(m/2)x^2 \pmod{P}$. Since $5|P$, it is obvious that $5|n$. Now we divide the proof into two cases.

Case 1 : Assume that $5|t$. Then $t = 5s$ for some positive integer s and therefore $n = mt = 5ms$. By (2.16), we obtain

$$U_n = U_{5ms} = U_{ms} ((P^2 + 4)^2 U_{ms}^4 + 5(P^2 + 4)U_{ms}^2 + 5) = 5U_m x^2. \quad (3.22)$$

Since ms is even and $5|P$, it is seen that $5|U_{ms}$ by 4. Also we have $(P^2 + 4)^2 U_{ms}^4 + 5(P^2 + 4)U_{ms}^2 + 5 = V_{ms}^4 - 3V_{ms}^2 + 1$ by (2.13). So rearranging the equation (3.22) gives

$$x^2 = (U_{ms}/U_m) ((V_{ms}^4 - 3V_{ms}^2 + 1)/5).$$

Clearly, $(U_{ms}/U_m, (V_{ms}^4 - 3V_{ms}^2 + 1)/5) = 1$. This implies that $U_{ms} = U_m a^2$ and $V_{ms}^4 - 3V_{ms}^2 + 1 = 5b^2$ for some positive integers a and b . Thus by Theorem 3.1, we get $V_{ms} = 2$, implying that $ms = 0$, which is impossible.

Case 2 : Assume that $5 \nmid t$. Since $5|n$, it follows that $5|m$. Then we can write $m = 5^r a$ with $5 \nmid a$, $2|a$, and $r \geq 1$. It can be seen by (2.16) that $U_m = U_{5^r a} = 5U_{5^{r-1}a}(5a_1 + 1)$ for some positive integer a_1 . And thus we conclude that $U_m = U_{5^r a} = 5^r U_a (5a_1 + 1)(5a_2 + 1) \dots (5a_r + 1)$ for some positive integers a_i with $1 \leq i \leq r$. Let $A = (5a_1 + 1)(5a_2 + 1) \dots (5a_r + 1)$. Then we have $U_m = 5^r U_a A$. In a similar manner, we get $U_n = U_{5^r at} = 5^r U_{at} (5b_1 + 1)(5b_2 + 1) \dots (5b_r + 1)$ for some

positive integers b_j with $1 \leq j \leq r$. Let $B = (5b_1 + 1)(5b_2 + 1)\dots(5b_r + 1)$. It is obvious that $5 \nmid B$. Thus we have $U_n = 5^r U_{at} B$. Substituting the new values of U_n and U_m into $U_n = 5U_m x^2$ gives

$$5^r U_{at} B = 5 \cdot 5^r U_a A x^2. \quad (3.23)$$

This shows that

$$U_{at} B = 5 U_a A x^2.$$

On the other hand, since a and at are even, it follows from Lemma 5 that $U_{at} \equiv (at/2)P \pmod{P^2}$ and $U_a \equiv (a/2)P \pmod{P^2}$. So (3.23) becomes

$$5^r (at/2)PB \equiv 5 \cdot 5^r (a/2)PAx^2 \pmod{P^2}.$$

Rearranging the above congruence gives

$$(at/2)B \equiv 5(a/2)Ax^2 \pmod{P}.$$

Since $5|P$, it follows that $5|(at/2)B$, implying that $5|atB$. This contradicts with the fact that $5 \nmid a$, $5 \nmid t$, and $5 \nmid B$. This completes the proof of Theorem 3.6. ■

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